Solutions: Homework 6

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Problem 1. (a) Prove Abel's Theorem: Let $\sum a_n(z-a)^n$ have radius of convergence 1 and suppose that $\sum a_n$ converges to A. Prove that

$$\lim_{r\to 1^-} \sum a_n r^n = A.$$

(b) Use Abel's theorem to prove that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - ...$

Proof. (a) WLOG, we can assume that $a = 0$. Let

$$s_N = \sum_{n=0}^{N} a_n$$

Then we know that $s_N \to A$ as $N \to \infty$. Note that $a_n = s_n - s_{n-1}$ for $n \geq 1$. Fix $r < 1$. Now,

$$\sum_{n=0}^{N} a_n r^n = a_0 + \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = a_0(1-r) + \sum_{n=1}^{N-1} s_n (r^n - r^{n+1}) + s_N r^N$$

for all $N \geq 1$. Taking the $1-r$ outside from the summation, we get

$$\sum_{n=0}^{N} a_n r^n = (1-r) \sum_{n=0}^{N-1} s_n r^n + s_N r^N.$$

Letting $N \to \infty$, we get

$$\sum a_n r^n = (1-r) \sum s_n r^n$$

So we just have to show that the RHS above converges to A as $r \to 1^-$. Let $\epsilon > 0$ be arbitrary. Let $M \in \mathbb{N}$ be such that $|s_n - A| \leq \epsilon/2$ for all $n \geq M$. Since $s_n \to A$, there exists $C > 0$ such that $|s_n - A| \leq C$ for all $n \geq 0$. We have

$$\left|(1-r) \sum s_n r^n - A\right| = \left|(1-r) \sum (s_n - A) r^n\right|$$

$$\leq (1-r) \sum_{n=0}^{M-1} |s_n - A| r^n + (1-r) \sum_{n=M}^{\infty} |s_n - A| r^n \leq C(1 - r^M) + \frac{\epsilon r^M}{2}$$

Choose r such that $1 - r^M \leq \frac{\epsilon}{2C}$. We then have

$$\left|(1-r)\sum s_n r^n - A\right| \leq \epsilon.$$

This completes the proof.
(b) We know that $\sum \frac{(-1)^{n+1}}{n}$ converges by the alternating test. Also, if $a_n = \frac{(-1)^n}{n}$, then $\limsup |a_n|^{1/n} = 1$, so the radius of convergence of the series $\sum a_n z^n$ is 1. Suppose that $A := 1 - \frac{1}{2} + \frac{1}{3} - ....$ Then by part (a),

$$\lim_{r \to 1^-} \sum \frac{(-1)^{n+1}}{n} r^n = A.$$

But we know that $\sum \frac{(-1)^n}{n+1} r^n = \log(1+r)$ for $-1 < r < 1$. So $A = \log 2$.

□

Problem 2. Let f be an entire function and suppose that there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

Proof. Since f is bounded in the compact set $\{|z| \leq R\}$, there exists M' such that $|f(z)| \leq M'$ for $|z| \leq R$. Let $k > n$. Let $R' > R$ be arbitrary. Then, by Cauchy's estimate, we have

$$|f^{(k)}(0)| \leq \frac{k! \max(M', M(R')^n)}{(R')^k}$$

Since this holds for all $R' > R$, taking the limit as $R' \to \infty$, we get $|f^{(k)}(0)| = 0$ because $k > n$. This implies that $f^{(k)}(0) = 0$ for all $k > n$. We know that f, being entire, has a power series expansion around 0, say $\sum a_n z^n$ with $a_k = \frac{1}{k!} f^{(k)}(0) = 0$ for all $k > n$. This implies that f is a polynomial of degree $\leq n$. □

Problem 3. Let $U : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all z in $\mathbb{C}$; prove that U is constant.

Proof. Since U is harmonic, we can find a V (harmonic conjugate) such that $f = U + iV$ is entire. Then the function $g = f + 1$ is also entire and never zero as its real part is always ≥ 1. So $\frac{1}{g}$ is also entire. But note that

$$\left|\frac{1}{g(z)}\right| = \frac{1}{\sqrt{(U(z)+1)^2 + V(z)^2}} \leq \frac{1}{U(z)+1} \leq 1$$

for all $z \in \mathbb{C}$. This implies that $\frac{1}{g}$, hence g, and hence f is constant, by Liouville's theorem. □

Problem 4. Show that the Integral Formula follows from Cauchy's Theorem.

Proof. Suppose that Cauchy's Theorem holds. Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, ..., \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + ... + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} \setminus G$, then we want to show that for $a \in G \setminus \cup_{k=1}^m \{\gamma_k\}$, we have

$$f(a) \sum_{k=1}^m n(\gamma_k; a) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz$$

Fix $a \in G \setminus \cup_{k=1}^m \{\gamma_k\}$. Define the function $g : G \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a. \end{cases}$$

Then g is analytic. We apply Cauchy's theorem to g. We then have

$$\sum_{k=1}^m \int_{\gamma_k} g = 0.$$

But we know that

$$\sum_{k=1}^m \int_{\gamma_k} g = \sum_{k=1}^m \int_{\gamma_k} \frac{f(z) - f(a)}{z-a} dz$$

$$= \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{z-a} dz - f(a) \sum_{k=1}^m \int_{\gamma_k} \frac{1}{z-a} dz$$

$$= \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{z-a} dz - 2\pi i f(a) \sum_{k=1}^m n(\gamma_k; a).$$

This completes the proof. □

Problem 5. Let G be a region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

Proof. Since analyticity is a local property, it is enough to prove that f is analytic on each open disk contained in G. So, WLOG, we assume $G = B(a; R)$. Let T be any triangular path in G. Then for $w \in \mathbb{C} \setminus G, n(T; w) = 0$ by Theorem 4.4. So, by Cauchy's theorem, we know that for all $n \geq 1$,

$$\int_T f_n = 0.$$

Now since a uniform limit of continuous functions is continuous, we know that f is continuous. We apply Lemma 2.7 to conclude that

$$\int_T f = 0.$$

Since T was arbitrary, by Morera's theorem, we know that f is analytic on G. □